Grading guide, Pricing Financial Assets, August 2018

1. Let the time t spot price on a commodity be S_t.
   (a) Two models for the commodity price are given:

$$d\ln S = (\theta(t) - a\ln S)dt + \sigma dz$$
$$dS = (\theta^\star(t) - a\ln S)Sdt + \sigma Sdz$$

   Use Ito's lemma to show that they are equivalent when $\theta(t) = \theta^\star(t) - \frac{1}{2}\sigma^2$

   (b) Interpret the model.
   (c) What is required for the above model of the commodity price to be a model of the price under a risk neutral equivalent martingale measure $\mathbb{Q}$?

Solution:

(a) Use Ito's lemma on the transformation $\ln$ of the second equation.

(b) One may take outset in the first formulation. The model of log prices has a drift rate with a time varying mean reversion. The drift rate of the log price is 0 when $S = \exp(\theta^\star(t) - a)$. The model of the log price has a constant volatility. No further interpretation is necessary, but you can use the flexibility of the θ-term to capture productivity gains in the production of the commodity, inflation (if nominal prices are considered) or seasonal variations.

(c) Under a risk neutral equivalent martingale measure the expected return of holding the commodity must be the risk free rate. Here we assume that storage is possible. This constitutes a satisfactory answer (as commodities are not explicitly dealt with in the syllabus).

But the analysis may be carried further. Assume the risk free rate to be $r(t)$, that storage is feasible, and that net storage costs (costs less convenience yield) can be described as a continuous proportional payment stream of $\delta(S,t)$, then we must have $\theta^\star(t) - a\ln S = r(t) + \delta(S,t)$. Thus the modelled drift rate may be interpreted as a model of the net storage costs.

2. Consider a one factor interest rate model with the following process for the instantaneous short rate r:

$$dr = m(r,t)dt + s(r,t)dz$$

   (a) View the price of a zero-coupon bond with expiry at date T as a function $P_T(r,t)$. Use Ito's lemma to find an expression for the process that $dP_T(r,t)$ follows.
   (b) Assume that this is modelled under a risk-neutral probability measure $\mathbb{Q}$. What will be the drift rate of P_T?
   (c) In some models of this form the price of the zero coupon bond can be expressed as

$$P_T(t,r) = A(t,T)\mathrm{e}^{-B(t,T)r}$$

   Show that $B(t,T)$ can be interpreted as a measure of the interest rate risk of the bond.

Solution:

(a) Using Ito's lemma we get (omitting the arguments):

$$dP_T = \left(\frac{\partial P_T}{\partial r} m + \frac{\partial P_T}{\partial t} + \frac{1}{2} \frac{\partial^2 P_T}{\partial r^2} s^2 \right) dt + \frac{\partial P_T}{\partial r} s dz$$

(b) The expected return must be the risk free rate, so the drift rate of the zero coupon price must be rP_T

(c) Note that

$$\frac{\partial P_T}{\partial r} = -BP_T$$

Thus B can be seen to measure the relative capital loss on the zero coupon bond from an increase in the short rate. B is the duration of the bond (Hull 8ed p 687).

3. Consider a Credit Default Swap (CDS).
   (a) Describe the instrument and its payment structure.
   (b) Consider a tranched CDS. Explain the payment structure and define the terms *attachment point* and *detachment point*.
   (c) Consider a tranched CDS on a large portfolio on underlying bonds (*names*). For a given average level of credit risk, e.g. expressed by the average credit spread on the underlying portfolio, explain how different levels of the assessed correlation of defaults of the issuers will influence the relative pricing of the tranches.

Solution:

(a)

Definition 0.1 (Credit Default Swap)**.** *A* ***Credit Default Swap*** *(CDS) is a contract between a protection seller and a protection buyer based on a specified Nominal Principal. The protection buyer pays a running premium (the CDS spread) until maturity of the contract or a Credit Event on the Reference Entity of the CDS. At a Credit Event the protection seller pays a Specified Amount (Fixed or (1-Recovery) times the Nominal Principal) to the protection buyer*

See Hull p. 549.

(b) The structure and terms are defined in Hull p. 560ff

(c) This discussion is covered in Hull section 24.9 p. 561. A low level of correlation will make it likely that the losses can be absorbed by the tranches low in the capital structure in most states of the world whereas a high correlation will make it more likely that even high tranches occasionally will be hit. Thus for a given average level of credit risk the equity tranche will benefit from an increase in the perceived correlation and the highest (super senior) tranche will suffer. The effect on mezzanine tranches can be ambiguous.